

Statistical mechanical analysis of the linear vector channel in digital communication

Koujin Takeda[†], Atsushi Hatabu^{†‡} and Yoshiyuki Kabashima[†]

[†] Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology, Yokohama 226-8502, Japan

[‡] System IP Core Research Laboratories, NEC Corporation, 1753 Shimonumabe, Nakahara, Kawasaki 211-8666, Japan

Abstract. A statistical mechanical framework to analyze linear vector channel models in digital wireless communication is proposed for a large system. The framework is a generalization of that proposed for code-division multiple-access systems in *Europhys. Lett.*, **76** (2006) 1193 and enables the analysis of the system in which the elements of the channel transfer matrix are statistically correlated with each other. The significance of the proposed scheme is demonstrated by assessing the performance of an existing model of multi-input multi-output communication systems.

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1. Introduction

In recent years, the number of objects to which statistical mechanical analysis can be applied has increased rapidly. The digital wireless communication system is one such example, and many intriguing studies in this field have revealed a strong relationship between telecommunication systems and statistical mechanics [1, 2].

The linear vector channel is one of the basic categories of wireless communication system. Code division multiple access (CDMA) [3], which is employed in third-generation cellular phone systems and wireless LANs, is a type of linear vector channel. In the general CDMA scenario, many users transmit discrete symbols that are modulated by random signature sequences using a single channel, and mixtures of user signals and noises are received at the other end of the channel. This indicates that the problem of simultaneously demodulating user signals from received signals can be mapped to a virtual spin system governed by random interactions. This problem has been successfully solved by techniques developed in statistical mechanics for disordered systems, and in particular by the replica method [4, 5, 6, 7, 8].

The multi-input multi-output (MIMO) system is another well-known example of a linear vector channel to which the statistical mechanical approach is applicable [9, 10, 11]. A MIMO system is composed of many transmit and receive antennas. In a general scenario, multiple input signals transmitted from transmit antennas are

received at the receive antennas, being linearly transformed to multiple output signals by a channel transfer matrix. In several preceding studies, channel transfer matrices are regarded as deterministic. However, the elements of such matrices vary with time in actual cases, which implies that handling the matrices as random is more realistic. In the simplest model, each element of the matrix could be regarded as an independent Gaussian variable with zero mean. Unfortunately, modeling of this type is inadequate for describing realistic MIMO systems in which correlations among the matrix elements are, in general, not negligible due to spatial proximity among transmit or receive antennas. For continuous inputs that are modeled as Gaussian variables, simple expressions of performance evaluation can still be obtained by using knowledge of random matrix theory [12]. However, such expressions cannot be applied directly to discrete inputs, which are usually used in digital communication. Therefore, developing a framework to analyze MIMO systems, and, more generally, linear vector channels of discrete inputs are demanded.

The purpose of the present article is to meet such a demand. Recently, the authors proposed a scheme to analyze CDMA systems under the assumption that a cross-correlation matrix of signature sequences can be regarded as a sample generated from a certain type of random matrix ensemble, which is characterized by an eigenvalue spectrum [13]. We herein generalize this scheme so as to be applicable to a wider class of linear vector channels.

The present paper is organized as follows. In Section 2, the linear vector channel models investigated herein are introduced. Section 3, in which a framework to analyze a given system is developed based on the replica method, is the main part of this article. The assumption of uniformity of transmitted signals generally guarantees that the average of the replicated Boltzmann weight depends on replicated vectors only through overlaps among the replicated and original vectors. This makes it possible to evaluate typical properties of the target system using a single function, which is referred to as $G(x)$ [14, 15]. In general, the analysis of typical property requires assessment of quenched averages, which implies that $G(x)$ should be evaluated as a quenched average utilizing the replica method. However, we will show that the assessment of this function can generally be reduced to the calculation of an annealed average due to a distinctive property underlying the evaluation of the average eigenvalue spectrum of the cross-correlation matrix for a given channel transfer matrix ensemble using the replica method if the eigenvalue spectrum of the ensemble is self-averaging. In Section 4, the significance of the framework is demonstrated by application to one of the typical MIMO models called the Kronecker model. Finally, Section 5 presents a summary of the present study.

2. Model definition

A linear vector channel is defined as a system in which an input vector composed of K components, $\mathbf{b} = (b_k)$ ($k = 1, 2, \dots, K$) (boldface denotes vector or matrix), is linearly

transformed by an $L \times K$ channel transfer matrix \mathbf{H} and is additively degraded by noise. For generality and simplicity, we assume that \mathbf{H} and \mathbf{b} are defined over the complex number field, and the channel noise is given as circularly symmetric complex additive white Gaussian noise, the variance of which is N_0 . We denote $\text{Re}(u)$ and $\text{Im}(u)$ as real and complex parts of a complex number u , respectively. $|u| = \sqrt{\text{Re}(u)^2 + \text{Im}(u)^2}$ denotes the absolute value of u . Under these assumptions and notations, the output vector

$$\mathbf{r} = \mathbf{H}\mathbf{b}^0 + \sqrt{N_0}\boldsymbol{\eta}, \quad (1)$$

is received by a receiver, where \mathbf{b}^0 denotes the input vector that is actually transmitted, and the components of the noise vector $\boldsymbol{\eta}$, η_l ($l = 1, 2, \dots, L$), independently obey circularly symmetric complex normal distributions $P(\eta) = \pi^{-1} \exp[-|\eta|^2] = \pi^{-1} \exp[-(\text{Re}(\eta)^2 + \text{Im}(\eta)^2)]$.

In the performance analysis shown below, \mathbf{H} is regarded as a sample from a certain random matrix ensemble, typical samples of which are *dense*. Namely, we assume that most elements of typical \mathbf{H} do not vanish. Since an elegant framework has been already established for Gaussian inputs [12], we focus on cases of discrete inputs in which input symbols are expressed as $b_k \in \{e^{2\pi i s/S} | s = 0, 1, 2, \dots, S-1\} \equiv \mathcal{A}_S$, where $i = \sqrt{-1}$. This expression corresponds to standard digital communication schemes of binary phase shift-keying (BPSK) and quadratic phase shift-keying (QPSK) for $S = 2$ and $S = 4$, respectively. We further assume that \mathbf{b} is encoded so as to be uniformly generated as $P(\mathbf{b}) = 1/S^K$ for optimizing communication performance.

After receiving \mathbf{r} the remaining task for the receiver is to infer the original vector \mathbf{b}^0 . The optimal inference scheme to minimize the component-wise probability of incorrect estimation, which is referred to as P_b , is constructed from the posterior distribution

$$P(\mathbf{b}|\mathbf{r}) = Z^{-1} \exp \left[-\frac{1}{N_r} |\mathbf{r} - \mathbf{H}\mathbf{b}|^2 \right], \quad (2)$$

as $\hat{b}_k = \text{argmax}_{b_k} \left\{ \sum_{\mathbf{b} \setminus b_k} P(\mathbf{b}|\mathbf{r}) \right\}$ in the hope that a model parameter of noise variance N_r is in agreement with the correct value N_0 . Here,

$$Z \equiv \sum_{\mathbf{b} \in \mathcal{A}_S^K} \exp \left[-\frac{1}{N_r} |\mathbf{r} - \mathbf{H}\mathbf{b}|^2 \right] \quad (3)$$

serves as the partition function, and \hat{b}_k denotes the estimate of b_k^0 , where \mathcal{A}_S^K denotes the k -th extension of symbol set \mathcal{A}_S .

3. Analytical scheme

3.1. Gauge transformation and Haar measure

Since the posterior distribution (2) depends on predetermined random variables \mathbf{H} , \mathbf{b}^0 , and $\boldsymbol{\eta}$, we resort to the replica method for assessing typical property of the linear vector channel. Thus, we first substitute equation (1) into equation (3) and perform gauge

transformation $(b_k^0)^* b_k^a \rightarrow \tau_k^a$ ($k = 1, 2, \dots, K; a = 1, 2, \dots, n$), where $*$ denotes the complex conjugate. This yields an expression of the replicate partition function (3) for $n = 1, 2, \dots$ as

$$Z^n = \sum_{\boldsymbol{\tau}^1, \boldsymbol{\tau}^2, \dots, \boldsymbol{\tau}^n \in \mathcal{A}_S^K} \exp \left[-\frac{1}{N_r} \sum_{a=1}^n |\mathbf{H} \text{diag}(\mathbf{b}^0)(\mathbf{1} - \boldsymbol{\tau}^a) + \sqrt{N_0} \boldsymbol{\eta}|^2 \right], \quad (4)$$

where $\mathbf{1}$ is a K -dimensional vector, all elements of which are unity and $\boldsymbol{\tau}^a = (\tau_k^a)$ ($k = 1, 2, \dots, K; a = 1, 2, \dots, n$). The diagonal matrix $\text{diag}(\mathbf{b}^0)$ is defined as $\text{diag}(\mathbf{b}^0) \equiv (\delta_{kj} b_k^0)$. Next, we average this expression with respect to \mathbf{b}^0 over the correct prior $P(\mathbf{b}^0) = 1/S^K$, fixing gauged vectors $\{\boldsymbol{\tau}^a\} = \{\boldsymbol{\tau}^1, \boldsymbol{\tau}^2, \dots, \boldsymbol{\tau}^n\}$. Here, we consider a property whereby vectors $\text{diag}(\mathbf{b}^0)(\mathbf{1} - \boldsymbol{\tau}^a)$ ($a = 1, 2, \dots, n$) are sampled isotropically in K -dimensional vector space under constraints of relative configuration

$$(\mathbf{1} - \boldsymbol{\tau}^a) \cdot (\mathbf{1} - \boldsymbol{\tau}^b) = (\mathbf{b}^0 - \mathbf{b}^a) \cdot (\mathbf{b}^0 - \mathbf{b}^b) = K(1 - m_a^* - m_b + q_{ab}), \quad (5)$$

for $a \neq b (= 1, 2, \dots, n)$, and

$$(\mathbf{1} - \boldsymbol{\tau}^a) \cdot (\mathbf{1} - \boldsymbol{\tau}^a) = (\mathbf{b}^0 - \mathbf{b}^a) \cdot (\mathbf{b}^0 - \mathbf{b}^a) = 2K(1 - m_a), \quad (6)$$

for $a = 1, 2, \dots, n$, where \cdot is defined as $\mathbf{x} \cdot \mathbf{y} = \sum_{k=1}^K x_k^* y_k$ for complex vectors $\mathbf{x} = (x_k)$ and $\mathbf{y} = (y_k)$ ($k = 1, 2, \dots, K$). $m_a = (1/K) \mathbf{1} \cdot \boldsymbol{\tau}^a = (1/K) \mathbf{b}^0 \cdot \mathbf{b}^a$ and $q_{ab} = (1/K) \boldsymbol{\tau}^a \cdot \boldsymbol{\tau}^b = (1/K) \mathbf{b}^a \cdot \mathbf{b}^b = q_{ba}^*$ ($a, b = 1, 2, \dots, n$). This implies that for any fixed set of $\{\boldsymbol{\tau}^a\}$, the average of the replicated Boltzmann weight $\exp[-1/(N_r) \sum_{a=1}^n |\mathbf{H} \text{diag}(\mathbf{b}^0)(\mathbf{1} - \boldsymbol{\tau}^a) + \sqrt{N_0} \boldsymbol{\eta}|^2]$ with respect to \mathbf{b}^0 is assessed as

$$\begin{aligned} & \frac{1}{S^K} \sum_{\mathbf{b}^0 \in \mathcal{A}_S^K} \exp \left[-\frac{1}{N_r} \sum_{a=1}^n |\mathbf{H} \text{diag}(\mathbf{b}^0)(\mathbf{1} - \boldsymbol{\tau}^a) + \sqrt{N_0} \boldsymbol{\eta}|^2 \right] \\ & \simeq \int \mathcal{D}\mathbf{U} \exp \left[-\frac{1}{N_r} \sum_{a=1}^n |\mathbf{H}\mathbf{U}(\mathbf{1} - \boldsymbol{\tau}^a) + \sqrt{N_0} \boldsymbol{\eta}|^2 \right], \end{aligned} \quad (7)$$

where $\mathcal{D}\mathbf{U}$ denotes the Haar measure of unitary matrix \mathbf{U} , which is normalized as $\int \mathcal{D}\mathbf{U} = 1$.

Equation (7) is justified by the following arguments. For a fixed typical pair of $\{\boldsymbol{\tau}^a\}$ and \mathbf{H} , which is dense, components of $\mathbf{H} \text{diag}(\mathbf{b}^0)(\mathbf{1} - \boldsymbol{\tau}^a)$, denoted as $t_l^a = \sum_{k,j} H_{lk} b_k^0 \delta_{kj} (1 - \tau_j^a) = \sum_k H_{lk} b_k^0 (1 - \tau_k^a)$, are composed of many independent random variables and, therefore, can be dealt with as complex Gaussian variables as a consequence of the central limit theorem when \mathbf{b}^0 is sampled from the uniform distribution $P(\mathbf{b}^0) = 1/S^K$. This means that statistical properties of t_l^a are fully characterized by only the first and second moments, which are determined by those of matrix components of $\text{diag}(\mathbf{b}^0)$. More precisely, the relevant moments are evaluated as $[t_l^a]_{\mathbf{b}^0} = 0$ and $[(t_l^a)^* t_m^b]_{\mathbf{b}^0} = \sum_{k,n} H_{lk}^* H_{mn} [(b_k^0)^* b_n^0]_{\mathbf{b}^0} (1 - \tau_k^a)^* (1 - \tau_n^b) = \sum_{k,n} H_{lk}^* H_{mn} \delta_{kn} (1 - \tau_k^a)^* (1 - \tau_n^b) = \sum_k H_{lk}^* H_{mk} (1 - \tau_k^a)^* (1 - \tau_k^b) \simeq (\sum_k H_{lk}^* H_{mk}) \times K^{-1} (\mathbf{1} - \boldsymbol{\tau}^a) \cdot (\mathbf{1} - \boldsymbol{\tau}^b)$ for $l, m = 1, 2, \dots, L$ and $a, b = 1, 2, \dots, n$, where $[\cdot]_{\mathbf{b}^0}$ represents average with respect to $P(\mathbf{b}^0) = 1/S^K$. The final replacement for the second moments $[(t_l^a)^* t_m^b]_{\mathbf{b}^0}$ is allowed as \mathbf{H} and $\{\boldsymbol{\tau}^a\}$ are statistically uncorrelated a priori. Here,

it is noteworthy that the identical moments are reproduced by substituting unitary matrix \mathbf{U} for $\text{diag}(\mathbf{b}^0)$ in conjunction with replacement of the Haar measure $\mathcal{D}\mathbf{U}$ with $P(\mathbf{b}^0) = 1/\mathcal{S}^K$. This validates equation (7), which will be supported by numerical experiments shown later herein as well.

3.2. G -functions and free energy

Next, averaging with respect to $\boldsymbol{\eta}$, we obtain

$$\begin{aligned} & \int_{\mathbb{C}^L} \frac{d\boldsymbol{\eta}}{\pi^L} \exp[-|\boldsymbol{\eta}|^2] \times \exp \left[-\frac{1}{N_r} \sum_{a=1}^n |\mathbf{H}\mathbf{U}(\mathbf{1} - \boldsymbol{\tau}^a) + \sqrt{N_0}\boldsymbol{\eta}|^2 \right] \\ &= \exp[\text{Tr} \mathbf{R}\mathbf{L}(n)], \end{aligned} \quad (8)$$

where $d\boldsymbol{\eta} = \prod_{l=1}^L d\text{Re}(\eta_l) \prod_{l=1}^L d\text{Im}(\eta_l)$, $\mathbf{R} = \mathbf{U}^\dagger (\mathbf{H}^\dagger \mathbf{H}) \mathbf{U}$ and

$$\begin{aligned} \mathbf{L}(n) = & -\frac{1}{N_r} \sum_{a=1}^n (\mathbf{1} - \boldsymbol{\tau}^a)(\mathbf{1} - \boldsymbol{\tau}^a)^\dagger \\ & + \frac{N_0}{N_r(N_r + nN_0)} \left(\sum_{a=1}^n (\mathbf{1} - \boldsymbol{\tau}^a) \right) \left(\sum_{b=1}^n (\mathbf{1} - \boldsymbol{\tau}^b) \right)^\dagger, \end{aligned} \quad (9)$$

where $\int_{\mathcal{X}}$ denotes integration over a certain support set \mathcal{X} . Next, integrating equation (8) over the Haar measure $\mathcal{D}\mathbf{U}$ in conjunction with taking average with respect to \mathbf{H} yields an expression of the averaged Boltzmann factor:

$$\begin{aligned} & \overline{\exp \left[-\frac{1}{N_r} \sum_{a=1}^n |\mathbf{H} \text{diag}(\mathbf{b}^0)(\mathbf{1} - \boldsymbol{\tau}^a) + \sqrt{N_0}\boldsymbol{\eta}|^2 \right]} \\ & \simeq \left[\int \mathcal{D}\mathbf{U} \exp[\text{Tr} \mathbf{R}\mathbf{L}(n)] \right]_{\mathbf{H}} \\ & \simeq \left[\exp \left[K \text{Tr} G_{\mathbf{H}} \left(\frac{\mathbf{L}(n)}{K} \right) \right] \right]_{\mathbf{H}} \simeq \exp \left[K \text{Tr} \left[G_{\mathbf{H}} \left(\frac{\mathbf{L}(n)}{K} \right) \right]_{\mathbf{H}} \right] \end{aligned} \quad (10)$$

$$\equiv \exp \left[K \text{Tr} G \left(\frac{\mathbf{L}(n)}{K} \right) \right], \quad (11)$$

for large K , where $\overline{\cdots}$ and $[\cdots]_{\mathbf{H}}$ denote the averages with respect to \mathbf{b}^0 , $\boldsymbol{\eta}$, and \mathbf{H} and to only \mathbf{H} , respectively. Transformation of equation (10) is valid if the eigenvalue spectrum of $\mathbf{H}^\dagger \mathbf{H}$ is self-averaging, *i.e.*, if the discrepancy between the eigenvalue spectrum of typical samples of $\mathbf{H}^\dagger \mathbf{H}$ and its average vanishes as $K, L \rightarrow \infty$ with keeping the load $\beta = K/L$ finite, as assumed hereinafter. A practical method to evaluate functions $G_{\mathbf{H}}(x)$ and $G(x)$ and the validation of equation (10) are discussed later herein.

Intrinsic permutation symmetry among replicas naturally leads to the replica symmetric (RS) ansatz. This implies that configurations characterized by $\mathbf{1} \cdot \boldsymbol{\tau}^a = \mathbf{b}^0 \cdot \mathbf{b}^a = Km$ ($a = 1, 2, \dots, n$) and $\boldsymbol{\tau}^a \cdot \boldsymbol{\tau}^b = \mathbf{b}^a \cdot \mathbf{b}^b = Kq$ ($a \neq b$) provide the most dominant contribution to the evaluation of $\overline{Z^n}$. Note that the RS ansatz requires obvious symmetry $\boldsymbol{\tau}^a \cdot \boldsymbol{\tau}^b = \boldsymbol{\tau}^b \cdot \boldsymbol{\tau}^a$, which enforces $q_{ab} = q_{ba} = q$ to be *real* at the saddle point for $\forall a, b = 1, 2, \dots, n$. Under this ansatz, the $K \times K$ matrix $\mathbf{L}(n)$ has three types of eigenvalues: $\lambda_1 = -K(N_r + nN_0)^{-1}(1 - q + n(1 - 2m + q))$, $\lambda_2 = -KN_r^{-1}(1 - q)$ and

$\lambda_3 = 0$, the numbers of degeneracy of which are 1, $n - 1$, and $K - n$, respectively. This indicates that equation (11) is evaluated as

$$\exp \left[K \left(G \left(-\frac{1-q+n(1-2m+q)}{N_r+nN_0} \right) + (n-1)G \left(-\frac{1-q}{N_r} \right) \right) \right]. \quad (12)$$

In addition, the RS ansatz offers the number of microscopic configurations that satisfy constraints (5) and (6) as

$$\text{Tr}_{\{\boldsymbol{\tau}^a\}} \prod_{a=1}^n \delta(\mathbf{1} \cdot \boldsymbol{\tau}^a - Km) \prod_{a>b} \delta(\boldsymbol{\tau}^a \cdot \boldsymbol{\tau}^b - Kq) \simeq \exp [KS_n(q, m)], \quad (13)$$

where

$$S_n(q, m) = \text{Extr}_{\hat{q}, \hat{m}} \left\{ \ln \left[\int_{\mathbb{C}} D\zeta \left(\sum_{\tau \in \mathcal{A}_S} \exp \left[\text{Re} \left((\sqrt{\hat{q}}\zeta^* + \hat{m})\tau \right) \right] \right)^n \right] - n\hat{m}m - \frac{n}{2}\hat{q} - \frac{n(n-1)}{2}\hat{q}q \right\}, \quad (14)$$

where $\text{Extr}_u\{\dots\}$ indicates the extremization of $\{\dots\}$ with respect to u and $D\zeta = (d\text{Re}(\zeta)d\text{Im}(\zeta)/2\pi) \exp[-|\zeta|^2/2]$ denotes the complex Gaussian measure, the variance of which is normalized to unity in each direction of the real and complex axes. Analytically continuing equations (12) and (13) from $n \in \mathbb{N}$ to $n \in \mathbb{R}$ yields an expression for assessing the configurational average of free energy as

$$\begin{aligned} \frac{1}{K} \overline{\ln Z} &= \lim_{n \rightarrow 0} \frac{1}{nK} \ln \overline{Z^n} \\ &= \text{Extr}_{m, q, \hat{m}, \hat{q}} \left\{ G \left(-\frac{1-q}{N_r} \right) + \left(-\frac{1-2m+q}{N_r} + \frac{N_0(1-q)}{N_r^2} \right) G' \left(-\frac{1-q}{N_r} \right) \right. \\ &\quad \left. - \hat{m}m - \frac{\hat{q}(1-q)}{2} + \int_{\mathbb{C}} D\zeta \ln \left(\sum_{\tau \in \mathcal{A}_S} \exp \left[\text{Re} \left((\sqrt{\hat{q}}\zeta^* + \hat{m})\tau \right) \right] \right) \right\}. \end{aligned} \quad (15)$$

The saddle-point solution yields the bit error rate for the demodulation strategy $\hat{b}_k = \text{argmax}_{b_k} \left\{ \sum_{\mathbf{b} \setminus b_k} P(\mathbf{b}|\mathbf{r}) \right\}$ as $P_b = \int_{\mathbb{C}} D\zeta \Theta_{\text{error}}(\zeta; \hat{q}, \hat{m})$, where $\Theta_{\text{error}}(\zeta; \hat{q}, \hat{m})$ vanishes if $\exp[\text{Re}((\sqrt{\hat{q}}\zeta^* + \hat{m})\tau)]$ is maximized by $\tau = 1$ among $\tau \in \mathcal{A}_S$, and unity, otherwise.

Note that both the RS assessment presented here and that of the replica symmetry breaking (RSB) ansatz are generally possible following the current framework. For instance, as shown in Appendix A, one can evaluate the free energy under the ansatz of one step replica symmetry breaking (1RSB) by dividing n replicated vectors $\{\boldsymbol{\tau}^a\}$ into n/x groups of identical size x and assuming that the correct saddle point is characterized by the following relative configuration: $\boldsymbol{\tau}^a \cdot \boldsymbol{\tau}^b = K$, $K(q + \Delta)$, and Kq for $a = b$, for the case in which $a \neq b$ but a and b belong to an identical group and otherwise, respectively. In the limit $n \rightarrow 0$, the 1RSB saddle-point equation is obtained as

$$\begin{aligned} \hat{\Delta} &= \frac{2}{xN_r} (G'_0 - G'_1), \quad \hat{q} = \frac{2N_0}{N_r^2} G'_1 - \frac{2A}{N_r} G''_1, \quad \hat{m} = \frac{2}{N_r} G'_1, \\ \Delta &= \int_{\mathbb{C}} D\zeta \left(\frac{\int_{\mathbb{C}} D\eta \Xi^x |\langle \tau \rangle_1|^2}{\int_{\mathbb{C}} D\eta \Xi^x} - \left| \frac{\int_{\mathbb{C}} D\eta \Xi^x \langle \tau \rangle_1}{\int_{\mathbb{C}} D\eta \Xi^x} \right|^2 \right), \\ q &= \int_{\mathbb{C}} D\zeta \left| \frac{\int_{\mathbb{C}} D\eta \Xi^x \langle \tau \rangle_1}{\int_{\mathbb{C}} D\eta \Xi^x} \right|^2, \\ m &= \int_{\mathbb{C}} D\zeta \frac{\int_{\mathbb{C}} D\eta \Xi^x \text{Re}(\langle \tau \rangle_1)}{\int_{\mathbb{C}} D\eta \Xi^x}. \end{aligned} \quad (16)$$

Here, $G'_1 = G'(-(1 - q + (x - 1)\Delta)/N_r)$, $G''_1 = G''(-(1 - q + (x - 1)\Delta)/N_r)$, $G'_0 = G'(-(1 - q - \Delta)/N_r)$, $A = -(1 - 2m + q)/N_r + (N_0/N_r^2)(1 - q + (x - 1)\Delta)$, $\Xi = \sum_{\tau \in \mathcal{A}_S} \exp \left[\text{Re} \left((\sqrt{\hat{\Delta}}\eta^* + \sqrt{\hat{q}}\zeta^* + \hat{m})\tau \right) \right]$, and $\langle \tau \rangle_1 = \Xi^{-1} \sum_{\tau \in \mathcal{A}_S} \tau \exp \left[\text{Re} \left((\sqrt{\hat{\Delta}}\eta^* + \sqrt{\hat{q}}\zeta^* + \hat{m})\tau \right) \right]$.

For $x \rightarrow 1$, a useful relation

$$\frac{\int_{\mathbb{C}} D\eta \Xi \langle \tau \rangle_1}{\int_{\mathbb{C}} D\eta \Xi} = \frac{\sum_{\tau \in \mathcal{A}_S} \tau \exp \left[\text{Re} \left((\sqrt{\hat{q}}\zeta^* + \hat{m})\tau \right) \right]}{\sum_{\tau \in \mathcal{A}_S} \exp \left[\text{Re} \left((\sqrt{\hat{q}}\zeta^* + \hat{m})\tau \right) \right]} \equiv \langle \tau \rangle_0, \quad (17)$$

implies that the saddle-point condition of equation (16) is governed by only four out of six variables, *i.e.*, \hat{q}, \hat{m}, q and m . Actually, the condition for determining the four variables in this case corresponds to the saddle-point equation of RS free energy (15), which implies that $x \rightarrow 1$ RSB analysis is generally reduced to that of RS. Nevertheless, a nontrivial result can still be obtained by investigating the behaviors of $\hat{\Delta}$ and Δ , which are subserviently determined from equation (16). This equation guarantees that $\hat{\Delta} = \Delta = 0$ always satisfies the saddle-point condition. However, for $x \rightarrow 1$ stability analysis indicates that a nontrivial solution of $\hat{\Delta} > 0$ and $\Delta > 0$ emerges if

$$\frac{2}{N_r^2} G'' \left(-\frac{1 - q}{N_r} \right) \int_{\mathbb{C}} D\zeta (1 - |\langle \tau \rangle_0|^2)^2 > 1, \quad (18)$$

which is in accordance with the de Almeida-Thouless (AT) condition signaling the local instability of the RS solution [16].

3.3. Equivalence between quenched and annealed averages in the assessment of $G(x)$

$G_{\mathbf{H}}(x)$ can be assessed by several formulae [17, 18, 19]. One formula uses the Stieltjes (or Cauchy) transformation of $\rho_{\mathbf{H}}(\lambda) = (1/K) \sum_{k=1}^K \delta(\lambda - \lambda_k)$, which is the eigenvalue spectrum of cross-correlation matrix $\mathbf{H}^\dagger \mathbf{H}$,

$$x = \int \frac{d\lambda \rho_{\mathbf{H}}(\lambda)}{\Lambda(x) - \lambda} \quad (19)$$

where $\lambda_1, \lambda_2, \dots, \lambda_K$ are eigenvalues of $\mathbf{H}^\dagger \mathbf{H}$ and are guaranteed to be real because $\mathbf{H}^\dagger \mathbf{H}$ is Hermitian. For given x , this relationship determines $\Lambda(x)$ implicitly, which is termed the Stieltjes inversion formula. Using $\Lambda(x)$, $G_{\mathbf{H}}(x)$ is assessed as

$$G_{\mathbf{H}}(x) = \int_0^x dt (\Lambda(t) - t^{-1}), \quad (20)$$

which is equivalent to the R -transformation known in free probability theory [20, 21].

Here, we describe a rather primitive approach. For this objective, we substitute $x\mathbf{e}\mathbf{e}^\dagger$ for $\mathbf{L}(n)$ in equation (8), where \mathbf{e} is a certain complex vector, the length of which is fixed as $|\mathbf{e}|^2 = K$. Note that the eigenvalues of this operator are Kx and zero, the degeneracies of which are 1 and $K - 1$, respectively. Integrating over $\mathcal{D}\mathbf{U}$ yields the following expression:

$$\exp [KG_{\mathbf{H}}(x)] = \exp \left[K \text{Tr} G_{\mathbf{H}} \left(\frac{x\mathbf{e}\mathbf{e}^\dagger}{K} \right) \right] = \int \mathcal{D}\mathbf{U} \exp [\text{Tr} \mathbf{R}(x\mathbf{e}\mathbf{e}^\dagger)]$$

$$= \frac{\int_{\mathbb{C}^K} d\mathbf{u} \delta(|\mathbf{u}|^2 - Kx) \exp [\mathbf{u}^\dagger (\mathbf{H}^\dagger \mathbf{H}) \mathbf{u}]}{\int_{\mathbb{C}^K} d\mathbf{u} \delta(|\mathbf{u}|^2 - Kx)}, \quad (21)$$

where we set $\mathbf{u} = \sqrt{x} \mathbf{U} \mathbf{e}$. Inserting $\delta(|\mathbf{u}|^2 - Kx) = \int_{-i\infty}^{+i\infty} d\Lambda \exp [-\Lambda(|\mathbf{u}|^2 - Kx)] / (2\pi i)$ and employing the saddle-point method with respect to the integration of Λ yields the following expression:

$$\begin{aligned} G_{\mathbf{H}}(x) &= \text{E}_{\Lambda}^{\text{extr}} \left\{ -\frac{1}{K} \ln \det |\Lambda - \mathbf{H}^\dagger \mathbf{H}| + \Lambda x \right\} - \ln x - 1, \\ &= \text{E}_{\Lambda}^{\text{extr}} \left\{ -\frac{1}{K} \sum_{k=1}^K \ln |\Lambda - \lambda_k| + \Lambda x \right\} - \ln x - 1, \\ &= \text{E}_{\Lambda}^{\text{extr}} \left\{ -\int d\lambda \rho_{\mathbf{H}}(\lambda) \ln |\Lambda - \lambda| + \Lambda x \right\} - \ln x - 1. \end{aligned} \quad (22)$$

In conjunction with equation (11), this indicates that $G(x)$ is offered as

$$G(x) = \text{E}_{\Lambda}^{\text{extr}} \left\{ -\int d\lambda \rho(\lambda) \ln |\Lambda - \lambda| + \Lambda x \right\} - \ln x - 1, \quad (23)$$

using average spectrum $\rho(\lambda) = [\rho_{\mathbf{H}}(\lambda)]_{\mathbf{H}}$ if a self-averaging property $\rho_{\mathbf{H}}(\lambda) \rightarrow \rho(\lambda)$ as $K, L \rightarrow \infty$ ($\beta = K/L \sim O(1)$) holds for typical samples of \mathbf{H} .

$\rho(\lambda)$ can be formally assessed as follows [22]. For this, we introduce a partition function of complex Gaussian spins as

$$\begin{aligned} Z_{\mathbf{H}}^{\text{Gauss}}(\lambda) &\equiv \int_{\mathbb{C}^K} d\mathbf{u} \exp [-\mathbf{u}^\dagger (\Lambda \mathbf{I}_K - \mathbf{H}^\dagger \mathbf{H}) \mathbf{u}] \\ &= \pi^K [\det(\Lambda \mathbf{I}_K - \mathbf{H}^\dagger \mathbf{H})]^{-1}, \end{aligned} \quad (24)$$

where \mathbf{u} is a K -dimensional complex vector and \mathbf{I}_K denotes a $K \times K$ identity matrix. The dispersion formula

$$\delta(\Lambda - \lambda) = \lim_{\epsilon \rightarrow +0} \frac{1}{\pi} \text{Im} \left(\frac{1}{\Lambda - \lambda + i\epsilon} \right) = -\frac{1}{\pi} \text{Im} \left(\frac{\partial}{\partial \Lambda} \ln(\Lambda - \lambda) \right), \quad (25)$$

indicates that $\rho(\lambda)$ can be assessed as

$$\rho(\Lambda) = \frac{1}{\pi} \text{Im} \left[\frac{\partial}{\partial \Lambda} \frac{1}{K} [\ln Z_{\mathbf{H}}^{\text{Gauss}}(\Lambda)]_{\mathbf{H}} \right], \quad (26)$$

where $K^{-1} [\ln Z_{\mathbf{H}}^{\text{Gauss}}(\Lambda)]_{\mathbf{H}}$ can be evaluated by the replica method.

For this evaluation, we assess the moments of $Z_{\mathbf{H}}^{\text{Gauss}}(\lambda)$ for $n \in \mathbb{N}$ with use of the saddle-point method, assuming that the saddle point is characterized by the Hermitian matrix $\mathbf{Q} = (Q_{ab}) = (K^{-1} \mathbf{u}^a \cdot \mathbf{u}^b)$ ($a, b = 1, 2, \dots, n$). This yields the following expression:

$$\begin{aligned} &\frac{1}{K} \ln [(Z_{\mathbf{H}}^{\text{Gauss}}(\Lambda))^n]_{\mathbf{H}} \\ &= \text{E}_{\mathbf{Q}, \mathbf{Q}}^{\text{extr}} \left\{ \frac{1}{K} \left[\ln \int_{\mathbb{C}^{nK}} \prod_{a=1}^n d\mathbf{u}^a \exp \left[-\sum_{a=1}^n (\mathbf{u}^a)^\dagger \left((\Lambda - \hat{Q}_{aa}) \mathbf{I}_K - \mathbf{H}^\dagger \mathbf{H} \right) \mathbf{u}^a \right. \right. \right. \\ &\quad \left. \left. \left. + \sum_{a \neq b} \hat{Q}_{ab} \mathbf{u}^a \cdot \mathbf{u}^b \right] \right]_{\mathbf{H}} - \text{Tr} \hat{\mathbf{Q}} \mathbf{Q} \right\}, \end{aligned} \quad (27)$$

where $\hat{\mathbf{Q}} = (\hat{Q}_{ab})$ denotes the conjugate matrix of \mathbf{Q} (Hermitian: does not indicate the Hermitian conjugate of \mathbf{Q}) to perform the saddle-point assessment. A distinctive property of equation (27) is that $\hat{\mathbf{Q}} = 0$ always satisfies the saddle-point condition for $\forall n \in \mathbb{N}$. This means that

$$\begin{aligned} \frac{1}{nK} \ln [(Z_{\mathbf{H}}^{\text{Gauss}}(\Lambda))^n]_{\mathbf{H}} &= \frac{1}{K} [-\ln \det (\Lambda \mathbf{I}_K - \mathbf{H}^\dagger \mathbf{H})]_{\mathbf{H}} + \ln \pi \\ &= \frac{1}{K} \ln [Z_{\mathbf{H}}^{\text{Gauss}}(\Lambda)]_{\mathbf{H}}, \end{aligned} \quad (28)$$

generally holds for the partition function and $\forall n \in \mathbb{N}$. Extending this expression analytically from $n \in \mathbb{N}$ to $n \in \mathbb{R}$ and taking $n \rightarrow 0$ yield a formula by which to evaluate $\rho(\lambda)$ using the annealed average of the partition function as

$$\rho(\Lambda) = \frac{1}{\pi} \text{Im} \left[\frac{\partial}{\partial \Lambda} \frac{1}{K} \ln [Z_{\mathbf{H}}^{\text{Gauss}}(\Lambda)]_{\mathbf{H}} \right]. \quad (29)$$

Let us insert this expression to equation (23) and perform partial integral. This operation and the dispersion formula (25) yield another expression of $G(x)$ as follows:

$$\begin{aligned} G(x) &= \text{E}_{\Lambda}^{\text{tr}} \left\{ \frac{1}{K} \ln [Z_{\mathbf{H}}^{\text{Gauss}}(\Lambda)]_{\mathbf{H}} + \Lambda x - \ln \pi \right\} - \ln x - 1 \\ &= \text{E}_{\Lambda}^{\text{tr}} \left\{ \frac{1}{K} \ln \int_{\mathbb{C}^K} \frac{d\mathbf{u}}{\pi^K} [\exp [-\mathbf{u}^\dagger (\Lambda \mathbf{I}_K - \mathbf{H}^\dagger \mathbf{H}) \mathbf{u}]]_{\mathbf{H}} + \Lambda x \right\} - \ln x - 1 \\ &= \frac{1}{K} \ln \left[\frac{\int_{\mathbb{C}^K} d\mathbf{u} \delta(|\mathbf{u}|^2 - Kx) \exp [\mathbf{u}^\dagger (\mathbf{H}^\dagger \mathbf{H}) \mathbf{u}]}{\int_{\mathbb{C}^K} d\mathbf{u} \delta(|\mathbf{u}|^2 - Kx)} \right]_{\mathbf{H}} \\ &= \frac{1}{K} \ln [\exp [KG_{\mathbf{H}}(x)]]_{\mathbf{H}}. \end{aligned} \quad (30)$$

Equations (11), (22), and (30) indicate that equivalence between annealed and quenched averages holds in the assessment of $G(x)$, which validates replacement in equation (10).

The current argument may be useful in assessing the typical performance of an ensemble of channels. Equations (19) and (20) can be used for evaluating the performance of a single sample of \mathbf{H} or for evaluating the performance of a channel ensemble, in which the eigenvalue spectrum is fixed [13, 21]. However, naive extension to the analysis of the typical performance of a channel ensemble along this direction generally requires taking configurational averages with respect to a certain distribution of \mathbf{H} *after* evaluating the sample-by-sample performance of \mathbf{H} using equations (19) and (20). From a practical viewpoint, this is not possible. However, equation (30) indicates that the typical performance of a channel ensemble can be evaluated using a single function $G(x)$ that characterizes the ensemble property if the eigenvalue spectra are self-averaging. This function can be assessed by the calculation of an annealed average, which, practically speaking, is in most cases much simpler than the calculation of the quenched averages. Formula (29) is useful as well because this equation can be used to numerically evaluate $\rho(\lambda)$ for a certain class of ensemble, for which analytical evaluation of $\rho(\lambda)$ is difficult. This is demonstrated in the next section.

4. Application

4.1. Kronecker model

Let us demonstrate the significance of the proposed framework by analyzing a certain channel ensemble. The ensemble that we focus on is termed the Kronecker model, which is a standard MIMO model [12, 23, 24].

In this model, the channel transfer matrix \mathbf{H} is represented as

$$\mathbf{H} = \sqrt{\mathbf{R}_r} \mathbf{Z} \sqrt{\mathbf{R}_t}, \quad (31)$$

where $\mathbf{Z} = (z_{lk})$ is an $L \times K$ random matrix, each component of which is independently sampled from an identical circularly symmetric complex Gaussian distribution $P(z) = L\pi^{-1} \exp[-L|z|^2]$. Here, \mathbf{R}_r is an $L \times L$ Hermitian matrix that represents the effect of spatial correlation among receivers and antennas. The $K \times K$ Hermitian matrix \mathbf{R}_t represents similar effects for transmit antennas. We assume that eigenvalue spectra of \mathbf{R}_r and \mathbf{R}_t are given as $\rho_r(\lambda)$ and $\rho_t(\lambda)$, respectively.

4.2. Average eigenvalue spectrum

For analyzing the typical property, we first introduce an expression of partition function of the complex Gaussian spins

$$\begin{aligned} Z_{\mathbf{H}}^{\text{Gauss}}(\Lambda) &= \int_{\mathbb{C}^K} d\mathbf{u} \exp[-\mathbf{u}^\dagger(\Lambda \mathbf{I}_K - \mathbf{H}^\dagger \mathbf{H})\mathbf{u}] \\ &= \int_{\mathbb{C}^K} d\mathbf{u} \exp\left[-\Lambda|\mathbf{u}|^2 + (\mathbf{Z}\sqrt{\mathbf{R}_t}\mathbf{u})^\dagger \mathbf{R}_r(\mathbf{Z}\sqrt{\mathbf{R}_t}\mathbf{u})\right]. \end{aligned} \quad (32)$$

Note that for fixed \mathbf{u} in the integrand of equation (32) $\mathbf{Z}\sqrt{\mathbf{R}_t}\mathbf{u} \equiv \mathbf{v} = (v_l)(l = 1, 2, \dots, L)$ can be handled as a circularly symmetric zero-mean L -dimensional complex Gaussian random vector that is characterized by covariance $[v_l^* v_j]_{\mathbf{H}} = L^{-1} \mathbf{u}^\dagger \mathbf{R}_t \mathbf{u} \delta_{lj} = K^{-1} \beta \mathbf{u}^\dagger \mathbf{R}_t \mathbf{u} \delta_{lj}$ ($l, j = 1, 2, \dots, L$) when each component of \mathbf{Z} in equation (31) is independently sampled from the identical distribution $P(z) = L\pi^{-1} \exp[-L|z|^2]$. Taking this property into consideration, we evaluate the annealed average of equation (32) as

$$\begin{aligned} [Z_{\mathbf{H}}^{\text{Gauss}}(\Lambda)]_{\mathbf{H}} &= \int d(KQ) d(K\beta T) \int_{\mathbb{C}^K} d\mathbf{u} \delta(|\mathbf{u}|^2 - KQ) \delta(\mathbf{u}^\dagger \mathbf{R}_t \mathbf{u} - K\beta T) \\ &\quad \times \exp[-\Lambda|\mathbf{u}|^2] \times \int_{\mathbb{C}^L} \frac{d\mathbf{v}}{(\pi\beta T)^N} \exp\left[-\frac{|\mathbf{v}|^2}{\beta T} + \mathbf{v}^\dagger \mathbf{R}_r \mathbf{v}\right] \\ &= \int d(KQ) d(K\beta T) \int_{-i\infty}^{i\infty} \frac{d\hat{Q}}{2\pi i} \frac{d\hat{T}}{2\pi i} \int_{\mathbb{C}^K} d\mathbf{u} \exp\left[-\mathbf{u}^\dagger((\Lambda + \hat{Q})\mathbf{I}_K - \hat{T}\mathbf{R}_t)\mathbf{u}\right] \\ &\quad \times \exp\left[K(\hat{Q}Q - \hat{T}T)\right] \times [\det(\mathbf{I}_L - \beta T \mathbf{R}_r)]^{-1} \\ &= \int d(KQ) d(K\beta T) \int_{-i\infty}^{i\infty} \frac{d\hat{Q}}{2\pi i} \frac{d\hat{T}}{2\pi i} \exp\left[K(\hat{Q}Q - \hat{T}T)\right] \\ &\quad \times \pi^K [\det((\Lambda + \hat{Q})\mathbf{I}_L - \hat{T}\mathbf{R}_t)]^{-1} \times [\det(\mathbf{I}_L - \beta T \mathbf{R}_r)]^{-1}. \end{aligned} \quad (33)$$

Utilizing the relations $K^{-1} \ln \det((\Lambda + \hat{Q})\mathbf{I}_K - \hat{T}\mathbf{R}_t) = \int d\lambda \rho_t(\lambda) \ln(\Lambda + \hat{Q} - \hat{T}\lambda)$ and $L^{-1} \ln \det(\mathbf{I}_L - \beta T\mathbf{R}_r) = \int d\lambda \rho_r(\lambda) \ln(1 - \beta T\lambda)$, this equation indicates that the annealed average of the partition function is evaluated by the saddle-point method as

$$\begin{aligned} & \frac{1}{K} \ln [Z_{\mathbf{H}}^{\text{Gauss}}(\Lambda)]_{\mathbf{H}} \\ &= \text{Extr}_{\hat{Q}, \hat{Q}, \hat{T}, T} \left\{ \ln \pi + \hat{Q}Q - \hat{T}T \right. \\ & \quad \left. - \int d\lambda \rho_t(\lambda) \ln(\Lambda + \hat{Q} - \hat{T}\lambda) - \frac{1}{\beta} \int d\lambda \rho_r(\lambda) \ln(1 - \beta T\lambda) \right\}. \end{aligned} \quad (34)$$

This yields the following equations:

$$T = \int d\lambda \frac{\lambda \rho_t(\lambda)}{\Lambda - \hat{T}\lambda}, \quad (35)$$

$$\hat{T} = \int d\lambda \frac{\lambda \rho_r(\lambda)}{1 - \beta T\lambda}, \quad (36)$$

which are only relevant for the saddle-point condition because $\hat{Q} = 0$ always holds. The solution of equations (35) and (36) yields the average eigenvalue spectrum as

$$\rho(\Lambda) = \frac{1}{\pi} \text{Im}(Q) = \frac{1}{\pi} \text{Im} \left\{ \int d\lambda \frac{\rho_t(\lambda)}{\Lambda - \hat{T}\lambda} \right\}. \quad (37)$$

4.3. Performance assessment for correlation matrices of Töplitz type

We applied the proposed scheme to the case in which both correlation matrices \mathbf{R}_r and \mathbf{R}_t are of the Töplitz type, in which the (i, j) elements of \mathbf{R}_r and \mathbf{R}_t are given as $r^{|i-j|}$ and $t^{|i-j|}$, respectively, where $0 < r < 1$ and $0 < t < 1$. For matrices of this type, eigenvalue spectra can be computed analytically in the limit of $K, L \rightarrow \infty$ as

$$\rho_r(\lambda) = \frac{1}{\pi \lambda \sqrt{(\alpha_+ - \lambda)(\lambda - \alpha_-)}}, \quad \alpha_{\pm} = \frac{1 \pm r}{1 \mp r}, \quad (38)$$

and similarly for $\rho_t(\lambda)$. For the purpose of illustrating generality, we employed numerical methods based on equations (23) and (29) to analyze this model, although expression (38) potentiates further analytical treatment, which will be reported elsewhere [25].

Figure 1 shows a comparison between the numerical average of the eigenvalue spectrum estimated from 200 samples of a 500×750 random channel transfer matrix (markers) and the theoretical prediction calculated from equation (37) by numerically solving equations (35) and (36) (curve). The excellent agreement between the experimental and theoretical data in this figure indicates that our approach to the assessment of the average eigenvalue spectrum based on the numerical saddle-point analysis of the annealed average of a partition function works rather efficiently in this case.

Figures 2(a) and 2(b) represent the communication performance assessed by the current framework. For convenience, the data in these figures are computed for $S = 2$ (BPSK) and *real* channels in which all elements of \mathbf{H} and channel noises are restricted to real numbers by replacing complex normal distribution $P(u) = \pi^{-1} \exp[-|u|^2]$

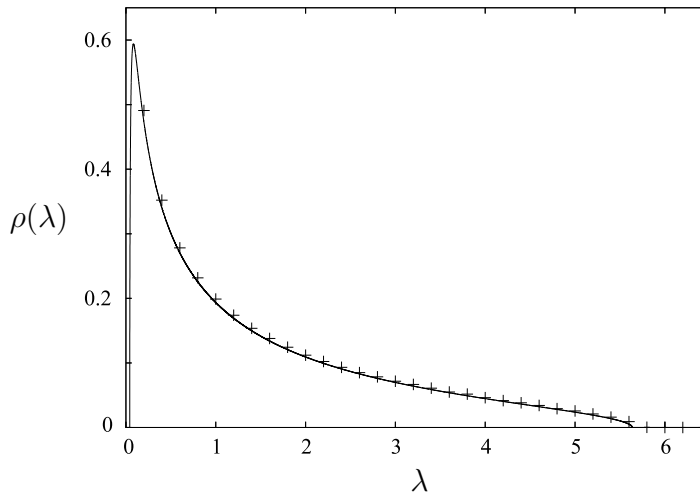


Figure 1. Eigenvalue spectrum of channel transfer cross-correlation matrix $\mathbf{H}^\dagger \mathbf{H}$. The comparison of the analytical result obtained by the scheme demonstrated in Section 4 (curve) and the result obtained by numerical diagonalization of randomly-generated cross-correlation matrix (markers) is depicted. Here, each parameter is set as $r = t = 0.2$, $\beta = 1.5$, and 200 samples of a 500×750 random channel transfer matrix are used for numerical diagonalization. The results show good agreement.

and unitary matrix \mathbf{U} with real normal distribution $P(x) = (2\pi)^{-1/2} \exp[-x^2/2]$ and orthogonal matrix \mathbf{O} in computing $G(x)$. Even if such a restriction is imposed, the framework is completely parallel. The only difference is that prefactor $1/2$ is placed in front of the definitions of $G_{\mathbf{H}}(x)$ and $G(x)$. The curves are provided by the saddle-point condition of equation (15), numerically assessing $G(x)$ from $\rho(\lambda)$ using equation (23), while the experimental results denoted by markers were obtained by the Thouless-Anderson-Palmer equation of the proposed system based on a strategy similar to equation (12) of ref. [13]. The reasonably good consistency between the curves and markers indicates the significance of the proposed framework in the performance assessment of linear vector channels of discrete inputs.

5. Summary

In summary, we have developed a framework by which to analyze linear vector channels of discrete inputs based on the replica method. Assuming uniformity of the prior probability of inputs makes it possible to characterize the typical property of the objective channel ensemble by a single function $G(x)$ if the eigenvalue spectrum of the cross-correlation matrix of the channel transfer matrix is self-averaging. We have also presented a practical scheme by which to evaluate $G(x)$ and an average eigenvalue spectrum $\rho(\lambda)$ using an annealed average of a partition function of Gaussian spins. The significance of the proposed scheme is demonstrated by application to an existing channel ensemble called the Kronecker model.

Future studies will include the mathematical validation of equation (7) and the

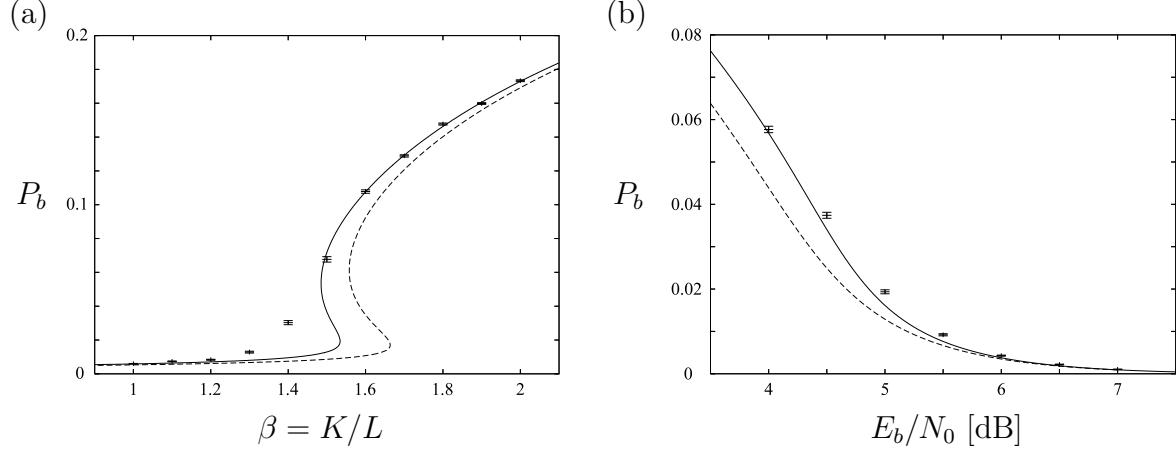


Figure 2. (a): Bit error rate P_b vs. the ratio $\beta = K/L$ under the condition $N_0 = N_r = 0.37, L = 1024$, and a 500 sample average. The solid/broken lines indicate the theoretical prediction obtained by the proposed scheme for the cases of $r = t = 0.2/r = t = 0$ (uncorrelated), respectively. The bars indicate the result of the numerical experiment in the case of $r = t = 0.2$, which shows good agreement with the prediction, except for the region in the vicinity of the waterfall (first-order phase transition) point. (b): Bit error rate P_b vs. signal-to-noise ratio E_b/N_0 under the condition of $\beta = 1.1, r = t = 0.2, L = 1024$, and a 500 sample average. The bars, which express the numerical results, show good agreement with the theoretical prediction (solid). The broken line indicates the analytical results for the uncorrelated case.

development of a practical demodulation algorithm [26, 27].

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Appendix A. Free energy under 1RSB ansatz

Under the one-step replica symmetry (1RSB) ansatz, n replicated replicas $\boldsymbol{\tau}^1, \boldsymbol{\tau}^2, \dots, \boldsymbol{\tau}^n$ are divided into n/x groups of identical size x , and the relevant saddle point is characterized by relative configuration

$$\boldsymbol{\tau}^a \cdot \boldsymbol{\tau}^b = \begin{cases} K, & a = b, \\ K(q + \Delta), & a \text{ and } b \text{ belong to an identical group,} \\ Kq, & \text{otherwise,} \end{cases} \quad (\text{A.1})$$

and $\mathbf{1} \cdot \boldsymbol{\tau}^a = m$ ($a, b = 1, 2, \dots, n$), where q, Δ , and m are assumed to be real to satisfy obvious symmetry $\boldsymbol{\tau}^a \cdot \boldsymbol{\tau}^b = \boldsymbol{\tau}^b \cdot \boldsymbol{\tau}^a$. Here, x serves as Parisi's replica symmetry breaking parameter after analytical continuation. For $\{\boldsymbol{\tau}^a\}$ that satisfies this configuration, $\mathbf{L}(n)$ has four types of eigenvalues: $\lambda_1 = -K(N_r + nN_0)^{-1}(1 - q + (x - 1)\Delta + n(1 - 2m + q))$,

$\lambda_2 = -KN_r^{-1}(1 - q + (x - 1)\Delta)$, $\lambda_3 = -KN_r^{-1}(1 - q - \Delta)$ and $\lambda_4 = 0$, the degeneracies of which are 1, $n/x - 1$, $n - n/x$, and $K - n$, respectively. This provides the following expression:

$$\begin{aligned} & \text{Tr} G \left(\frac{\mathbf{L}(n)}{K} \right) \\ &= nG \left(-\frac{1 - q - \Delta}{N_r} \right) + \frac{n}{x} \left(G \left(-\frac{1 - q + (x - 1)\Delta}{N_r} \right) - G \left(-\frac{1 - q - \Delta}{N_r} \right) \right) \\ &+ G \left(-\frac{1 - q + (x - 1)\Delta + n(1 - 2m + q)}{N_r + nN_0} \right) - G \left(-\frac{1 - q + (x - 1)\Delta}{N_r} \right). \end{aligned} \quad (\text{A.2})$$

Using the saddle-point method, the number of microscopic configurations that satisfy the current ansatz increases as $\exp [KS_n^{\text{1RSB}}(q, \Delta, m; x)]$, where

$$\begin{aligned} & S_n^{\text{1RSB}}(q, \Delta, m; x) \\ &= \text{Extr}_{\hat{q}, \hat{\Delta}, \hat{m}} \left\{ \ln \left[\int_{\mathbb{C}} D\zeta \left(\int_{\mathbb{C}} D\eta \left(\text{Tr}_{\tau \in A_S} \exp \left[\text{Re} \left((\sqrt{\hat{\Delta}}\eta^* + \sqrt{\hat{q}}\zeta^* + \hat{m})\tau \right) \right] \right)^x \right)^{\frac{n}{x}} \right] \right. \\ &\quad - n\hat{m}m - \frac{n}{2}(\hat{q} + \hat{\Delta}) \\ &\quad \left. - \frac{nx(x-1)}{2} \left((\hat{q} + \hat{\Delta})(q + \Delta) - \hat{q}q \right) - \frac{n(n-1)}{2}\hat{q}q \right\}. \end{aligned} \quad (\text{A.3})$$

Equations (A.2) and (A.3) indicates that under the 1RSB ansatz, free energy is assessed as

$$\begin{aligned} & \frac{1}{K} \overline{\ln Z} = \lim_{n \rightarrow 0} \frac{1}{nK} \ln \overline{Z^n} \\ &= \text{Extr}_{q, \Delta, m, \hat{q}, \hat{\Delta}, \hat{m}} \left\{ G \left(-\frac{1 - q - \Delta}{N_r} \right) \right. \\ &\quad + \frac{1}{x} \left(G \left(-\frac{1 - q + (x - 1)\Delta}{N_r} \right) - G \left(-\frac{1 - q - \Delta}{N_r} \right) \right) \\ &\quad + \left(-\frac{1 - 2m + q}{N_r} + \frac{N_0(1 - q + (x - 1)\Delta)}{N_r^2} \right) G' \left(-\frac{1 - q + (x - 1)\Delta}{N_r} \right) \\ &\quad - \hat{m}m - \frac{1}{2}(\hat{q} + \hat{\Delta}) - \frac{x-1}{2} \left((\hat{q} + \hat{\Delta})(q + \Delta) - \hat{q}q \right) + \frac{1}{2}\hat{q}q \\ &\quad \left. + \frac{1}{x} \int_{\mathbb{C}} D\zeta \ln \left[\int_{\mathbb{C}} D\eta \left(\text{Tr}_{\tau \in A_S} \exp \left[\text{Re} \left((\sqrt{\hat{\Delta}}\eta^* + \sqrt{\hat{q}}\zeta^* + \hat{m})\tau \right) \right] \right)^x \right] \right\}. \end{aligned} \quad (\text{A.4})$$

- [1] Guo D and Verdú S 2005 *IEEE Trans. on Infor. Theory* **51** 1983
- [2] Nishimori H 2001 *Statistical Physics of Spin Glasses and Information Processing - An Introduction* (Oxford: Oxford University Press)
- [3] See e.g. Verdú S *Multuser Detection* 1998 (Cambridge: Cambridge University Press)
- [4] Tanaka T 2001 *Europhys. Lett* **54** 540
- [5] Tanaka T 2002 *IEEE Trans. on Infor. Theory* **48** 2888
- [6] Wen C -K, Lee Y -N, Chen J -T and Ting P 2005 *IEEE Trans. on Signal Processing* **53** 2059
- [7] Wen C -K, Ting P and Chen J -T 2006 *IEEE Trans. on Comm.* **54** 349
- [8] Guo D 2006 *IEEE Trans. on Infor. Theory* **52** 1765
- [9] Moustakas A L, Simon S H and Sengupta A M 2003 *IEEE Trans. on Infor. Theory* **49** 2545

- [10] Müller R R 2003 *IEEE Trans. on Signal Processing* **51** 2821
- [11] Takeuchi K, Tanaka T and Yano T arXiv:0706.3170
- [12] Tulino A M and Verdú S 2004 *Random Matrix Theory and Wireless Communications* (Hanover, MA: Now Publishers)
- [13] Takeda K, Uda S and Kabashima Y 2006 *Europhys. Lett.* **76** 1193
- [14] Marinari E, Parisi G and Ritort F 1994 *J. Phys. A: Math. Gen.* **27** 7647
- [15] Parisi G and Potters M 1995 *J. Phys. A: Math. Gen.* **28** 5267
- [16] de Almeida J R L and Thouless D J 1978 *J. Phys. A: Math. Gen.* **11** 983
- [17] Oppor M and Winther O 2001 *Phys. Rev. Lett.* **86** 3695
- [18] Oppor M and Winther O 2001 *Phys. Rev. E* **64** 056131
- [19] Cherrier R, Dean D S and Lefèvre A 2003 *Phys. Rev. E* **67** 046112
- [20] Voiculescu D V, Dykema K J and Nica A 1992 *Free Random Variables* (Providence, R.I.: American Mathematical Society)
- [21] Müller R R, Guo D and Moustakas A L arXiv:0706.1169
- [22] Oppor M 1989 *Europhys. Lett.* **8** 389
- [23] Shiu D -S, Foschini G J, Gans M J and Kahn J M 2000 *IEEE Trans. on Communications* **48** 502
- [24] Chizhik D, Rashid-Farrokhi F, Ling J and Lozano A 2000 *IEEE Communications Letters* **4** 337
- [25] Hatabu A, Takeda K and Kabashima Y work in progress
- [26] Kabashima Y 2003 *J. Phys. A: Math. Gen.* **36** 11111
- [27] Tanaka T and Okada M 2005 *IEEE Trans. on Infor. Theory* **51** 700